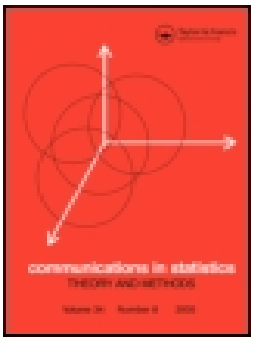

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

A new extension of the Fréchet distribution: Properties and its characterization

Zohdy M. Nofal & M. Ahsanullah

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
https://doi.org/10.1080/03610926.2018.1465080

A new extension of the Fréchet distribution: Properties and its characterization

Zohdy M. Nofal[a] and M. Ahsanullah[b]

[a]Department of Statistics, Mathematics and Insurance, Benha University, Banha, Egypt; [b]Department of Management Sciences, Rider University, Lawrenceville, NJ, USA

ABSTRACT
A new lifetime model, which extends the Fréchet distribution called the generalized transmuted Fréchet distribution is proposed and studied. Various of its structural properties including ordinary and incomplete moments, generating function, residual and reversed residual lifes, order statistics and probability weighted moments are derived. Two characterization theorems are presented. The maximum likelihood method is used to estimate the model parameters. The flexibility of the new distribution is illustrated using a real data set. It can serve as an alternative model to other lifetime models available in the literature for modeling positive real data in many areas.

ARTICLE HISTORY
Received 7 January 2017
Accepted 21 February 2018

KEYWORDS
Fréchet model; generalized transmuted-G family; order statistics' maximum likelihood; moments; characterization

MATHEMATICS SUBJECT CLASSIFICATIONS
62No1; 62No2; 62E10

1. Introduction

Recently, there has been an increased interest among statisticians to develop new extended distributions to be more capable for modeling data in different areas such as lifetime analysis, engineering, economics, finance, demography, actuarial, biological, and medical sciences.

The Fréchet (Fr) distribution is an important distribution developed within the extreme value theory. It has applications in life testing, floods, horse racing, rainfall, queues in supermarkets, sea waves, and wind speeds. Further information about the Fr distribution and its applications can be explored in Kotz and Nadarajah (2000).

Aiming a more flexible Fr distribution, for many years statisticians have been developing various extensions and modified forms of the Fr distribution, with different number of parameters. For example, the exponentiated Fr due to Nadarajah and Kotz (2003), the beta Fr due to Nadarajah and Gupta (2004) and Barreto-Souza, Cordeiro, and Simas (2011), the transmuted Fr due to Mahmoud and Mandouh (2013), the gamma extended Fr due to da Silva et al. (2016), the Marshall–Olkin Fr due to Krishna et al. (2013), the Kumaraswamy Fr due to Mead and Abd-Eltawab (2014), the transmuted Marshall–Olkin Fr due to Afify et al. (2015), the Kumaraswamy Marshall–Olkin Fr due to Afify et al. (2016a), the Kumaraswamy transmuted Marshall–Olkin Fr due to Yousof et al. (2016), the Weibull Fr due to Afify et al. (2016b) and the beta exponential Fr due to Mead et al. (2017).

The probability density function (PDF) and cumulative distribution function (CDF) of the Fr distribution are given by (for $x \geq 0$)

CONTACT Zohdy M. Nofal dr.zohdynofal@fcom.bu.edu.eg Department of Statistics, Mathematics and Insurance, Benha University, Banha, 13511, Egypt.
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/lsta.

$$g(x;\alpha,\beta) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \text{ and } G(x;\alpha,\beta) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \tag{1}$$

respectively, where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter.

In this paper, we define and study a new lifetime model called the *generalized transmuted Fréchet* (GTFr) distribution. Its main feature is that three additional shape parameters are inserted in Equation (1) to provide greater flexibility for the generated distribution. Based on the *generalized transmuted-G* (GT-G) family of distributions, we construct the new five-parameter GTFr model and give a comprehensive description of some of its mathematical properties hoping that it will attract wider applications in engineering, survival and lifetime data, reliability, and other areas of research.

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative functions of the baseline model with parameter vector ξ. Nofal et al. (2017) defined the CDF of their GT-G family by

$$F(x;\lambda,a,b,\xi) = G(x;\xi)^{a}\left\{1+\lambda-\lambda G(x;\xi)^{b}\right\} \tag{2}$$

The PDF of the GT-G family is given by

$$f(x;\lambda,a,b,\xi) = g(x;\xi)\,G(x;\xi)^{a-1}\left\{a(1+\lambda)-\lambda(a+b)\,G(x;\xi)^{b}\right\} \tag{3}$$

where $a > 0, b > 0$ and $|\lambda| \le 1$ are shape parameters.

Henceforth, let G be a continuous baseline distribution. We define the GT-G distribution with three extra parameters a, b and λ by the PDF (Equation 3). A random variable X with PDF (Equation 3) is denoted by $X \sim$GT-G(λ, a, b, ξ). If $a = b = 1$, it corresponds to the transmuted class (TC) studied by Shaw and Buckley (2007). For $b = 0$, the GT-G family reduces to the exponentiated-G (E-G) family defined by Gupta et al. (1998) and finally the GT-G family reduces to the baseline distribution when $a = b = 1$ and $\lambda = 0$.

The rest of the paper is outlined as follows. In Section 2, we define the GTFr distribution, provide its sub-models and give some plots for its PDF and hazard rate function (HRF). We derive useful mixture representations for its PDF in Section 3. We provide in Section 4 some mathematical properties of the GTFr distribution including ordinary and incomplete moments, moments of the residual and reversed residual lifes, moment generating function (mgf), order statistics and probability weighted moments (PWMs). In Section 5, we provide some useful characterization results. The maximum likelihood estimates (MLEs) of the unknown parameters are obtained in Section 6. In Section 7, the GTFr distribution is applied to a real data set to illustrate its potentiality. Finally, in Section 8, we provide some concluding remarks.

2. The GTFr distribution

By inserting the CDF in Equation (1) in Equation (2), we obtain the CDF of the GTFr model (for $x > 0$)

$$F(x) = \exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+\lambda-\lambda\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\} \tag{4}$$

The corresponding PDF of Equation (4) is given by

$$f(x) = \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{a(1+\lambda)-\lambda(a+b)\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\} \tag{5}$$

where α, β, a, and b are positive parameters and $|\lambda| \leq 1$. Here, α is a scale parameter representing the characteristic life and β, a, b, and λ are shape parameters representing the different patterns of the GTFr distribution. We denote a random variable X having PDF (Equation 5) by $X \sim \text{GTFr}(\alpha, \beta, \lambda, a, b)$. The HRF, reversed hazard rate function and cumulative hazard rate function of X are given by

$$h(x) = \frac{\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{a(1+\lambda) - \lambda(a+b)\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}{1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+\lambda-\lambda\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}$$

$$\tau(x) = \frac{\beta\alpha^{\beta}x^{-\beta-1}\left\{a(1+\lambda) - \lambda(a+b)\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}{\left\{1+\lambda-\lambda\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}$$

and

$$H(x) = -\ln\left(1-\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+\lambda-\lambda\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\right)$$

respectively.

The GTFr distribution is a very flexible model having several special cases. Its 11 sub-models are listed in Table 1. The plots of the GTFr density for some parameter values α, β, λ, a, and b are displayed in Figures 1 and 2. The plots in Figures 1 and 2 show that the pdf of the GTFr distribution can be reversed J-shape, right-skewed, left-skewed or bimodal.

Figures 3 and 4 provides some plots of the hrf of the GTFr model for selected parameter values. It can be seen that the hrf is very flexible so the proposed distribution should be useful to model increasing, decreasing, unimodal, and bathtub failure rates behavior.

3. Linear representation

The GTFr density function Equation (5) can be expressed as

$$f(x) = a(1+\lambda)g(x)G(x)^{a-1} - \lambda(a+b)g(x)G(x)^{a+b-1} \tag{6}$$

By inserting Equation (1) in Equation (6), we obtain

$$f(x) = a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Table 1. Sub-models of the GTFr model.

Model	α	β	λ	a	b	Author
GTIR	α	2	λ	a	b	New
GTIEx	α	1	λ	a	b	New
TFr	α	β	λ	1	1	Mahmoud and Mandouh (2013)
TIEx	α	1	λ	1	1	–
TIR	α	2	λ	1	1	–
EFr	α	β	0	a	0	Nadarajah and Kotz (2003)
EIR	α	2	0	a	0	–
EIEx	α	1	0	a	0	–
Fr	α	β	0	1	1	Fréchet (1924)
IR	α	2	0	1	1	Keller and Kamath (1982)
IEx	α	1	0	1	1	Treyer (1964)

Abbreviations: IR, Inverse Rayleigh; IEx, Inverse Exponential; E, Exponentiated; T, Transmuted.

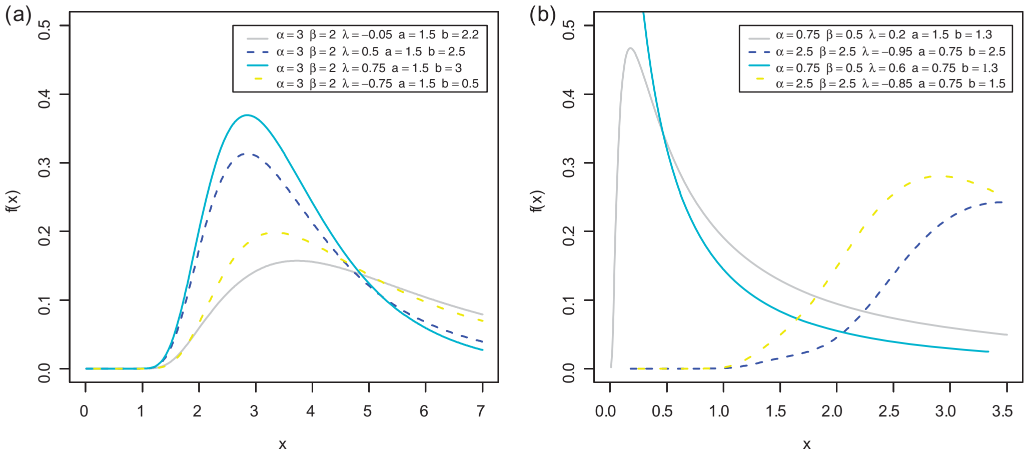

Figure 1. The PDF plots of the GTFr distribution for some parameter values.

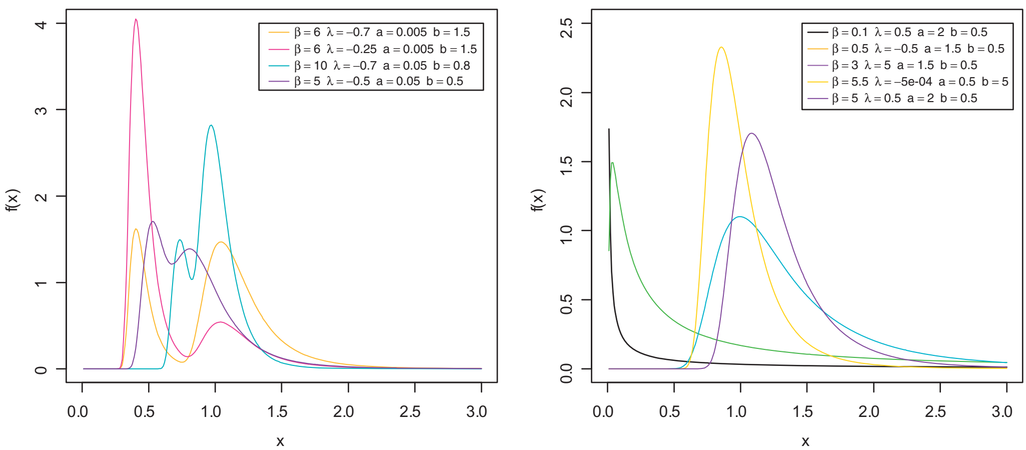

Figure 2. The PDF plots of the GTFr distribution for some parameter values.

$$-\lambda\,(a+b)\,\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-\,(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right] \tag{7}$$

Then, the PDF in Equation (7) can be expressed as

$$f(x) = (1+\lambda)\;h_a(x) - \lambda\,h_{a+b}(x) \tag{8}$$

where $h_j(x), j = a, a+b$, is the Fr density with scale parameter $\alpha j^{1/\beta}$ and shape parameter β.

Equation (8) is a mixture of two Fr densities. As a result some of the properties that we have obtained can also be obtained considering this property.

Let Z be a random variable having the Fr distribution Equation (1) with parameters α and β. For $r < \beta$, the rth ordinary and incomplete moments of Z are, respectively, given by $\mu_r' = \alpha^r\,\Gamma(1 - r/\beta)$ and $\varphi_r(t) = \alpha^r\gamma(1 - r/\beta, (\alpha/t)^{\beta})$, where

$$\Gamma(m) = \int_0^{\infty} y^{m-1}\,e^{-y}dy \;\;\text{and}\;\; \gamma(s,z) = \int_0^{z} y^{s-1}\,e^{-y}dy$$

are the complete gamma function and lower incomplete gamma function, respectively.

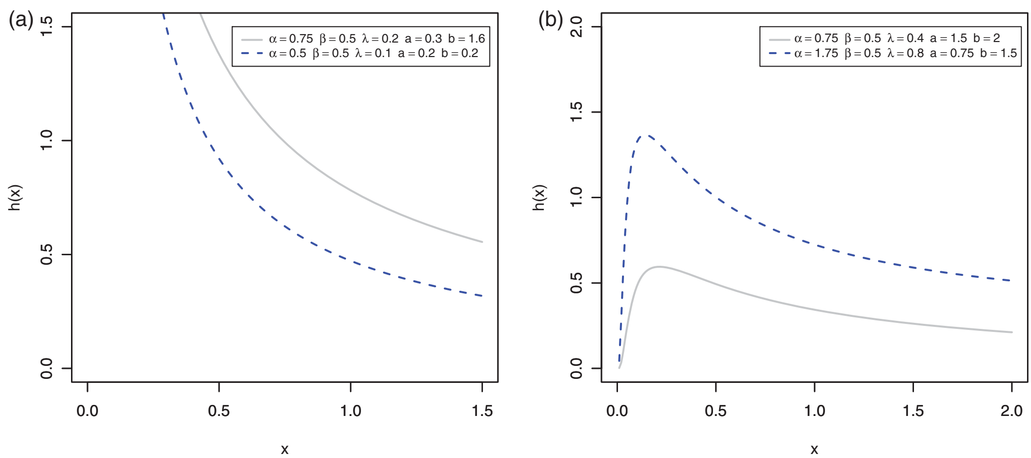

Figure 3. The hrf plots of the GTFr distribution for some parameter values.

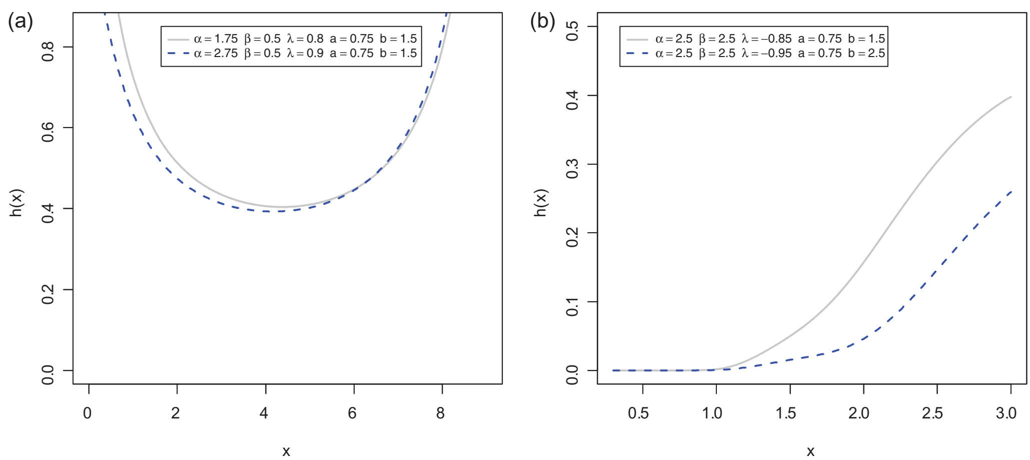

Figure 4. The hrf plots of the GTFr distribution for some parameter values.

4. Properties

Here, we investigate mathematical properties of the GTFr distribution including ordinary and incomplete moments, moment of the residual and reversed residual lifes, mgf, order statistics, and PWMs.

4.1. Ordinary and incomplete moments

The nth ordinary moment of X is given by

$$\mu'_n = E(X^n) = (1+\lambda)\, E\left(Y_a^n\right) - \lambda E\left(Y_{a+b}^n\right)$$

where

$$E\left(Y_j^n\right) = \int_0^\infty x^n\, h_j\,(x)\, dx, j = a, a+b$$

Therefore, for $n < \beta$, we obtain

$$\mu'_n = (1+\lambda)\, a^{\frac{n}{\beta}} \alpha^n\, \Gamma\left(1-\frac{n}{\beta}\right) - \lambda\,(a+b)^{\frac{n}{\beta}}\, \alpha^n\, \Gamma\left(1-\frac{n}{\beta}\right) \tag{9}$$

Setting $n = 1$ in Equation (9), we have the mean of X.

Remark 1. *Table 2 shows that different choices of a and b could yield positive and negative skewness.*

The nth central moment of X is given by

$$M_n = E(X-\mu)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\mu'_1)^n\, \mu'_{n-k}$$

The nth incomplete moment of the GTFr distribution is defined by $\varphi_n(t) = \int_0^t x^n f(x)dx$. Using Equation (8), we can write

$$\varphi_n(t) = (1+\lambda)\int_0^t x^n\, h_a(x)\, dx - \lambda \int_0^t x^n h_{a+b}(x)\, dx$$

and then using the lower incomplete gamma function, we obtain (for $n < \beta$)

$$\varphi_n(t) = (1+\lambda)\, a^{\frac{n}{\beta}} \alpha^n\, \gamma\left(1-\frac{n}{\beta}, a\left(\frac{\alpha}{t}\right)^{\beta}\right) - \lambda\,(a+b)^{\frac{n}{\beta}}\, \alpha^n\, \gamma\left(1-\frac{n}{\beta}, (a+b)\left(\frac{\alpha}{t}\right)^{\beta}\right)$$

The important application of the first incomplete moment, which follows from the above equation with $n = 1$, is related to the Lorenz and Bonferroni curves. These curves are very useful in economics, reliability, demography, insurance and medicine.

Another application of the first incomplete moment is related to the mean residual life and the mean waiting time given by $m_1(t) = [1-\varphi_1(t)]/R(t) - t$ and $M_1(t) = t - \varphi_1(t)/F(t)$, respectively.

Further, the amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean and about the median of X can be expressed as

$$\delta_\mu(X) = \int_0^\infty \left|X-\mu'_1\right| f(x)dx = 2\mu'_1 F(\mu'_1) - 2\varphi_1(\mu'_1)$$

and

$$\delta_M(X) = \int_0^\infty |X-M|\, f(x)dx = \mu'_1 - 2\varphi_1(M)$$

Table 2. Skewness of the GTFr model for various values of a and b.

a	b	Skewness	a	b	Skewness	a	b	Skewness
5	5	2.068135	2	2	−0.496087	4.00	5	1.202875
5	3	2.082621	2	1	−0.577528	3.00	5	0.302502
5	1	1.851670	2	0.9	−1.429561	2.00	5	−0.648299
4	4	1.242962	2	0.8	−1.428610	1.00	5	−1.783897
4	2	1.192853	2	0.7	−1.429425	2.90	5	0.209642
3	3	0.389904	2	0.6	−1.432128	2.69	5	0.012828
3	1	0.245648	2	0.5	−1.436843	2.68	5	0.003397

respectively, where $\mu_1' = E(X)$ comes from Equation (9), $F(\mu_1')$ is simply calculated from Equation (4), $\varphi_1(\mu_1')$ is the first incomplete moment and M is the median of X.

4.2. Residual and reversed residual life functions

The sth moment of the residual life, say $m_s(t) = E[(X-t)^s \mid X > t], t > 0,\ s = 1, 2, \ldots,$ uniquely determine $F(x)$. The sth moment of the residual life of X is given by

$$m_s(t) = \frac{1}{R(t)} \int_t^\infty (x-t)^s dF(x)$$

Using the generalized binomial series and Equation (8), we can write (for $r < \beta$)

$$m_s(t) = \frac{1}{R(t)} \sum_{r=0}^{s} (-t)^{s-r} \binom{s}{r} \left\{ (1+\lambda)\, a^{\frac{r}{\beta}} \alpha^r\, \Gamma\left(1 - \frac{r}{\beta}, a\left(\frac{\alpha}{t}\right)^\beta\right) - \lambda\,(a+b)^{\frac{r}{\beta}}\, \alpha^r\, \Gamma\left(1 - \frac{r}{\beta}, (a+b)\left(\frac{\alpha}{t}\right)^\beta\right) \right\}$$

where $\Gamma(k, z) = \int_z^\infty y^{k-1}\, e^{-y} dy$ is the the upper incomplete gamma function.

An interesting function called the mean residual life (MRL) function or the life expectation at age t is defined by $m_1(t) = E\left[(X-t) \mid X > t\right]$, which represents the expected additional life length for a unit which is alive at age t. Setting $s = 1$ in the last equation gives the MRL of X.

The sth moment of the reversed residual life uniquely determines $F(x)$ and it is defined by

$$M_s(t) = E\left[(t-X)^s \mid X \le t\right] = \frac{1}{F(t)} \int_0^t (t-x)^s dF(x)$$

where $t > 0$ and $s = 1, 2, \ldots$

Therefore, the sth moment of the reversed residual life of X, given that $r < \beta$, becomes

$$M_s(t) = \frac{1}{F(t)} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \left\{ (1+\lambda)\, a^{\frac{r}{\beta}} \alpha^r\, \gamma\left(1 - \frac{r}{\beta}, a\left(\frac{\alpha}{t}\right)^\beta\right) - \lambda\,(a+b)^{\frac{r}{\beta}}\, \alpha^r\, \gamma\left(1 - \frac{r}{\beta}, (a+b)\left(\frac{\alpha}{t}\right)^\beta\right) \right\}$$

The mean reversed residual life, also called mean inactivity time (MIT) or mean waiting time (MWT), represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$ and it is given by $M_1(t) = E[(t-X) \mid X \le t]$, and its simply follows from the last equation with $s = 1$.

4.3. Generating function

Let $M_j(t)$ be the mgf of Y_j, $j = a, a+b$. Therefore, using Equation (8) the mgf of X, say $M(t) = E(e^{tx})$, is given by

$$M(t) = (1+\lambda)\, M_a(x) - \lambda M_{a+b}(x) \tag{10}$$

First, we provide the mgf of the Fr distribution as provided by Afify et al. (2016b). We can write the mgf of $W = 1/X$ as

$$M(t; \alpha, \beta) = \beta \alpha^\beta \int_0^\infty e^{t/w}\, w^{(\beta-1)}\, e^{-(\alpha w)^\beta} dw$$

By expanding the first exponential and determining the integral, we obtain

$$M(t;\alpha,\beta)=\sum_{n=0}^{\infty}\frac{\alpha^{n}\,t^{n}}{n!}\,\Gamma\left(\frac{\beta-n}{\beta}\right)$$

Consider the Wright generalized hypergeometric function defined by

$${}_{p}\Psi_{q}\left[\begin{array}{c}(\gamma_1,A_1),\ldots,(\gamma_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{array};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\gamma_j+A_j\,n\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_j\,n\right)}\,\frac{x^{n}}{n!}$$

Then, we can write $M(t;\alpha,\beta)$ as

$$M(t;\alpha,\beta)={}_{1}\Psi_{0}\left[\begin{array}{c}\left(1,-\beta^{-1}\right)\\-\end{array};\alpha\,t\right]\tag{11}$$

Combining the Equations (10) and (11), the mgf of X reduces to

$$\begin{aligned}M(t)=\Bigg\{&(1+\lambda)\ {}_{1}\Psi_{0}\left[\begin{array}{c}\left(1,-\beta^{-1}\right)\\-\end{array};\alpha a^{1/\beta}\,t\right]\\&-\lambda\ {}_{1}\Psi_{0}\left[\begin{array}{c}\left(1,-\beta^{-1}\right)\\-\end{array};\alpha\,(a+b)^{1/\beta}\ t\right]\Bigg\}\end{aligned}$$

4.4. Order statistics

Let $X_1,\ldots,X_n$ denote n independent and identically distributed GTFr random variables. Further, let $X_{(1)},\ldots,X_{(n)}$ denote the order statistics from these n variables. Then, the PDF of the ith order statistic $X_{(i)}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x)=\frac{f(x)}{\mathrm{B}(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}F(x)^{i+j-1}$$

Using Equation (4), we obtain

$$\begin{aligned}F(x)^{i+j-1}&=\exp\left[-a\left(i+j-1\right)\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1+\lambda-\lambda\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{i+j-1}\\&=\sum_{k=0}^{\infty}(-1)^{k}\,(1+\lambda)^{i+j-1}\left(\frac{\lambda}{1+\lambda}\right)^{k}\binom{i+j-1}{k}\\&\quad\times\exp\left\{-\left[a\left(i+j-1\right)+kb\right]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\end{aligned}$$

Then, we have

$$\begin{aligned}f(x)F(x)^{i+j-1}=&\sum_{k=0}^{\infty}(-1)^{k}\,(1+\lambda)^{i+j-1}\left(\frac{\lambda}{1+\lambda}\right)^{k}\binom{i+j-1}{k}\\&\times\Bigg(a\,(1+\lambda)\,d\exp\left\{-\left[a\left(i+j\right)+kb\right]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\\&-\lambda\,(a+b)\,d\exp\left\{-\left[a\left(i+j\right)+b\,(k+1)\right]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\Bigg)\end{aligned}\tag{12}$$

where $d = \beta\alpha^{\beta}x^{-\beta-1}$. Then, we can write

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\left[a_{k,j}\, g_{a(i+j)+kb}\,(x) - b_{k,j}\, g_{a(i+j)+b(k+1)}\,(x)\right] \tag{13}$$

where

$$a_{k,j} = \frac{(-1)^{k+j}\, a\,(1+\lambda)\,(1+\lambda)^{i+j-1}}{\mathrm{B}(i, n-i+1)\left[a\left(i+j\right)+kb\right]}\left(\frac{\lambda}{1+\lambda}\right)^{k}\binom{n-i}{j}\binom{i+j-1}{k}$$

$$b_{k,j} = \frac{(-1)^{k+j}\,\lambda\,(a+b)\,(1+\lambda)^{i+j-1}}{\mathrm{B}(i, n-i+1)\left[a\left(i+j\right)+b\,(k+1)\right]}\left(\frac{\lambda}{1+\lambda}\right)^{k}\binom{n-i}{j}\binom{i+j-1}{k}$$

and $g_{\eta}\,(x)$ denotes the Fr density function with with scale parameter $\alpha\eta^{1/\beta}$ and shape parameter β. Thus, the density function of the GTFr order statistics is a mixture of Fr densities. Based on Equation (13), we can obtain some structural properties of $X_{i:n}$ from those Fr properties.

The nth moment of $X_{i:n}$ (for $n < \beta$) is given by

$$E(X_{i:n}^{n}) = \sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\left[a_{k,j}\, E\left(Y_{a(i+j)+kb}^{n}\right) - b_{k,j}E\left(Y_{a(i+j)+b(k+1)}^{n}\right)\right] \tag{14}$$

Equation (14) reveals that the nth moment of $X_{i:n}$ can be expressed as an infinite linear combination of Fr moments.

Based upon the moments in Equation (14), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable GTFr distributions. The L-moments are given by

$$\xi_{s} = \frac{1}{s}\sum_{k=0}^{s-1}(-1)^{k}\binom{s-1}{k}E\left(X_{s-k:s}\right),\; s \geq 1$$

4.5. Probability weighted moments

The PWMs are expectations of certain functions of a random variable. The PWM approach can be used for estimating parameters of any distribution whose inverse form cannot be expressed explicitly.

The (s, r)th PWM of X is defined by

$$\rho_{s,r} = E\left\{X^{s}\, F(X)^{r}\right\} = \int_{-\infty}^{\infty} x^{s}\, F(x)^{r} f\,(x)\; dx$$

Using Equation (12), we can write

$$\begin{aligned} f(x)F(x)^{r} = \sum_{k=0}^{\infty}(-1)^{k}\,(1+\lambda)^{r}\left(\frac{\lambda}{1+\lambda}\right)^{k}\binom{r}{k} \\ \times\left(a\,(1+\lambda)\, d\exp\left\{-\left[a\,(r+1)+kb\right]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\right. \\ \left. -\,\lambda\,(a+b)\, d\exp\left\{-\left[a\,(r+1)+b\,(k+1)\right]\left(\frac{\alpha}{x}\right)^{\beta}\right\}\right) \end{aligned}$$

or, equivalently, we have

$$f(x)F(x)^r = \sum_{k=0}^{\infty} \left[m_k \, g_{a(r+1)+kb}\,(x) - w_k \, g_{a(r+1)+b(k+1)}\,(x) \right]$$

where

$$m_k = \frac{(-1)^{k+j}\ a\,(1+\lambda)^{r+1}}{a\,(r+1)+kb} \left(\frac{\lambda}{1+\lambda}\right)^k \binom{r}{k}$$

and

$$w_k = \frac{(-1)^{k+j}\ \lambda\,(a+b)\,(1+\lambda)^{r}}{a\,(r+1)+b\,(k+1)} \left(\frac{\lambda}{1+\lambda}\right)^k \binom{r}{k}$$

Therefore, $\rho_{s,r}$ can be defined as an infinite linear combination of Fr moments, by

$$\rho_{s,r} = \sum_{k=0}^{\infty} \left[m_k E\left(Y^s_{a(r+1)+kb}\right) - w_k E\left(Y^s_{a(r+1)+b(k+1)}\right) \right]$$

where

$$E\left(Y^s_\delta\right) = \int_0^\infty x^s\, h_\delta\,(x)\,dx$$

Therefore, for $s < \beta$, we obtain

$$\rho_{s,r} = \sum_{k=0}^{\infty} \left\{ m_k \left[a\,(r+1)+kb\right]^{\frac{s}{\beta}} \alpha^s\, \Gamma\left(1-\frac{s}{\beta}\right) \right.$$
$$\left. -w_k \left[a\,(r+1)+b\,(k+1)\right]^{\frac{s}{\beta}} \alpha^s\, \Gamma\left(1-\frac{s}{\beta}\right) \right\}$$

5. Characterization

We will present two characterization theorems based on truncated moments. To prove the main characterization theorems we need the following two lemmas.

Assumption 1. *Suppose the random variable X is absolutely continuous with CDF $F(x)$ and PDF $f(x)$. Let*

$$\gamma = \sup\{x|F(x) > 0\} \quad \text{and} \quad \delta = \inf\{x|F(x) < 1\}$$

We assume $E(X)$ exist.

Lemma 1. *Let X be a random variable under the Assumption 1. Assume*

$$E(X|X \le x) = g(x)\tau(x)$$

where $g(x)$ is a continuous differentiable function with the condition

$$\int_\gamma^x \frac{u - g'(u)}{g(u)} du < \infty \text{ for all } x, \gamma < x < \delta$$

and

$$\tau(x) = \frac{f(x)}{F(x)}$$

Then

$$f(x) = c\, e^{\int_\gamma^x \frac{u - g'(u)}{g(u)} du}, \; \gamma < x < \delta$$

Proof. We have

$$g(x) = \frac{\int_\gamma^x ufu)du}{f(x)} \quad \text{or} \quad \int_\gamma^x ufu)du = g(x)f(x)$$

Differentiating with respect to x, we obtain

$$xf(x) = f'(x)g(x) + f(x)g'(x)$$

and hence

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}$$

On integrating both sides of the equation, we have

$$f(x) = c\, e^{\int_\gamma^x \frac{u - g'(u)}{g(u)} du}$$

and c is such that

$$\int_\gamma^\delta f(x)dx = 1$$

□

Lemma 2. *Let X be a random variable under the Assumption 1. Assume*

$$E(X|X \geq x) = g(x)h(x)$$

where $g(x)$ is a continuous differentiable function with the condition

$$\int_\gamma^x \frac{u + g'(u)}{g(u)} du < \infty, \; \textit{for all} \; x, \gamma < x < \delta$$

and

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Then

$$f(x) = c\, e^{-\int_\gamma^x \frac{u + g'(u)}{g(u)} du}, \gamma < x < \delta$$

where c is determinated by the condition $\int_\gamma^\delta f(x)dx = 1$.

Proof of this lemma is similar to the proof of Lemma 1.

Theorem 1. *Let X be a random variable satisfying the Assumption 1 with $\gamma = 0$ and $\delta = \infty$, then*

$$E(X|X \leq x) = g(x)\tau(x)$$

where

$$\tau(x) = \frac{f(x)}{F(x)}$$

$$g(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma^c_{a(\frac{\alpha}{t})^\beta}(1-\frac{1}{\beta}) - \lambda(a+b)^{1/\beta}\alpha\Gamma^c_{(a+b)(\frac{\alpha}{t})^\beta}(1-\frac{1}{\beta})}{a(1+\lambda)\beta\alpha^\beta x^{-\beta-1}\exp\left[-a(\frac{\alpha}{x})^\beta\right] - \lambda(a+b)\beta\alpha^\beta x^{-\beta-1}\exp\left[-(a+b)(\frac{\alpha}{x})^\beta\right]}, \beta > 1$$

with

$$\Gamma^c_x(\gamma) = \int_x^\infty u^{\gamma-1}e^{-u}du$$

If and only if

$$f(x) = a(1+\lambda)\beta\alpha^\beta x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^\beta\right] - \lambda(a+b)\beta\alpha^\beta x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^\beta\right]\beta > 1$$

Proof. If

$$f(x) = a(1+\lambda)\beta\alpha^\beta x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^\beta\right] - \lambda(a+b)\beta\alpha^\beta x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^\beta\right]$$

Then

$$\begin{aligned} f(x)g(x) &= \int_0^x \left\{a(1+\lambda)\beta\alpha^\beta u^{-\beta}\exp\left[-a\left(\frac{\alpha}{u}\right)^\beta\right] - \lambda(a+b)\beta\alpha^\beta u^{-\beta}\exp\left[-(a+b)\left(\frac{\alpha}{u}\right)^\beta\right]\right\} du \\ &= (1+\lambda)\alpha a^{1/\beta}\Gamma^c_{a(\frac{\alpha}{t})^\beta}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma^c_{(a+b)(\frac{\alpha}{t})^\beta}(1-1/\beta), \beta > 1 \end{aligned}$$

with

$$\Gamma^c_x(\gamma) = \int_x^\infty u^{\gamma-1}e^{-u}du$$

Thus

$$g(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma^c_{a(\frac{\alpha}{t})^\beta}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma^c_{(a+b)(\frac{\alpha}{t})^\beta}(1-1/\beta)}{a(1+\lambda)\beta\alpha^\beta x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^\beta\right] - \lambda(a+b)\beta\alpha^\beta x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^\beta\right]}$$

Suppose

$$g(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma^c_{a(\frac{\alpha}{t})^\beta}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma^c_{(a+b)(\frac{\alpha}{t})^\beta}(1-1/\beta)}{a(1+\lambda)\beta\alpha^\beta x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^\beta\right] - \lambda(a+b)\beta\alpha^\beta x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^\beta\right]}$$

Then we have

$$g\prime(x) = x - g(x)\left\{\frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}$$

where

$$A(x) = \frac{a}{x^{\beta+2}}\alpha^{\beta}\beta e^{-a\left(\frac{\alpha}{x}\right)^{\beta}}(\lambda+1)\left[a\beta\left(\frac{\alpha}{x}\right)^{\beta}-(\beta+1)\right]$$

and

$$B(x) = \frac{1}{x^{\beta+2}}\alpha^{\beta}\beta\lambda\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right](a+b)\left[\beta+1-(a+b)\beta\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Thus

$$\frac{x-g(x)}{g(x)} = \frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

By Lemma 1, we have

$$\frac{f'(x)}{f(x)} = \frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

Integrating the above equation, we obtain

$$f(x) = c\left\{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\right\}$$
$$\times\ \beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Using the boundary condition $\int_o^{\infty} f(x)dx = 1$, we will have $c = 1$. □

Theorem 2. *Let X be a random variable satisfying the Assumption 1 with $\gamma = 0$ and $\delta = \infty$, then*

$$E(X|X \geq x) = m(x)h(x)$$

where

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$m(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma_{a(\frac{\alpha}{t})^{\beta}}(1-1/\beta)-\lambda(a+b)^{1/\beta}\alpha\Gamma_{(a+b)(\frac{\alpha}{t})^{\beta}}(1-1/\beta)}{ca(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]},$$
$$\beta > 1$$

and

$$\Gamma_x(\gamma) = \int_0^x u^{\gamma-1}e^{-u}du$$

If and only if

$$f(x) = a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] - \lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right],$$

$$\beta > 1$$

Proof. If

$$f(x) = a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] - \lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Then

$$\begin{aligned} f(x)m(x) &= \int_x^{\infty}\left\{a(1+\lambda)\beta\alpha^{\beta}u^{-\beta}\exp\left[-a\left(\frac{\alpha}{u}\right)^{\beta}\right]\right. \\ &\quad \left. -\lambda(a+b)\beta\alpha^{\beta}u^{-\beta}\exp\left[-(a+b)\left(\frac{\alpha}{u}\right)^{\beta}\right]\right\}du \\ &= (1+\lambda)\alpha a^{1/\beta}\Gamma_{a(\frac{\alpha}{t})^{\beta}}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma_{(a+b)(\frac{\alpha}{t})^{\beta}}(1-1/\beta), \beta > 1 \end{aligned}$$

with

$$\Gamma_x(\gamma) = \int_0^x u^{\gamma-1}e^{-u}du$$

Thus

$$m(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma_{a(\frac{\alpha}{t})^{\beta}}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma_{(a+b)(\frac{\alpha}{t})^{\beta}}(1-1/\beta)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] - \lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

Suppose

$$m(x) = \frac{(1+\lambda)\alpha a^{1/\beta}\Gamma_{a(\frac{\alpha}{t})^{\beta}}(1-1/\beta) - \lambda(a+b)^{1/\beta}\alpha\Gamma_{(a+b)(\frac{\alpha}{t})^{\beta}}(1-1/\beta)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] - \lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1-}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

Then we have

$$g\prime(x) = -x - g(x)$$

$$\left\{\frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right] - \lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right\}$$

where

$$A(x) = \frac{a}{x^{\beta+2}}\alpha^{\beta}\beta e^{-a(\frac{\alpha}{x})^{\beta}}(\lambda+1)\left[a\beta\left(\frac{\alpha}{x}\right)^{\beta} - (\beta+1)\right]$$

and

$$B(x) = \frac{1}{x^{\beta+2}}\alpha^{\beta}\beta\lambda\,(a+b)\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]\left[\beta+1-(a+b)\beta\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Thus

$$-\frac{x+g(x)}{g(x)}$$

$$= \frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

By Lemma 2

$$\frac{f'(x)}{f(x)} = \frac{A(x)+B(x)}{a(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]}$$

Integrating the above equation, we obtain

$$f(x) = ca(1+\lambda)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]-\lambda(a+b)\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(a+b)\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

Using the boundary condition $\int_o^{\infty} f(x)dx = 1$, we will have $c = 1$. □

6. Estimation

The maximum likelihood estimators (MLEs) have desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. We consider the maximum likelihood to estimate the unknown parameters of the GTFr model from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of the GTFr distribution with unknown parameter vector $\phi = (\alpha, \beta, \lambda, a, b)^T$.

The log-likelihood function for ϕ, say $\ell = \ell(\phi)$, is given by

$$\ell = n\ln\beta + n\beta\ln\alpha - (\beta+1)\sum_{i=1}^{n}\ln(x_i) - a\sum_{i=1}^{n}s_i + \sum_{i=1}^{n}\ln(k_i)$$

where $s_i = (\alpha/x_i)^{\beta}$ and $k_i = \left[a(1+\lambda) - \lambda(a+b)\exp(-bs_i)\right]$.

The score vector is given by

$$\mathbf{U}(\phi) = \frac{\partial\ell}{\partial\phi} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}\right)^T$$

where

$$\frac{\partial\ell}{\partial\alpha} = \frac{n\beta}{\alpha} - \frac{a\beta}{\alpha}\sum_{i=1}^{n}s_i + \frac{\lambda b(a+b)\beta}{\alpha}\sum_{i=1}^{n}s_i\exp(-bs_i)$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + n\ln\alpha - \sum\nolimits_{i=1}^{n}\ln(x_i) - a\sum_{i=1}^{n}s_i\ln(\alpha/x_i)$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\frac{1}{k_i}\left[a-(a+b)\exp(-bs_i)\right]$$

$$\frac{\partial\ell}{\partial a} = -\sum_{i=1}^{n}s_i + \sum_{i=1}^{n}\frac{1}{k_i}\left[1+\lambda-\lambda\exp(-bs_i)\right]$$

and

$$\frac{\partial\ell}{\partial b} = \sum_{i=1}^{n}\frac{1}{k_i}\left[\lambda\left[(a+b)s_i+1\right]\exp(-bs_i)\right]$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\phi}) = 0$. By solving these equations simultaneously gives the MLEs $\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{a}$ and $\widehat{b}$. Statistical software can be used to solve these equations numerically by means of iterative techniques such as the Newton-Raphson algorithm because they can not be solved analytically. For the GTFr distribution all the second order derivatives exist.

For interval estimation of the model parameters, we require the 5×5 observed information matrix $J(\phi) = \{J_{rs}\}$ for $r, s = \alpha, \beta, \lambda, a, b$, whose elements are given in the Appendix. Under standard regularity conditions, the multivariate normal $N_5(0, J(\widehat{\phi})^{-1})$ distribution can be used to construct approximate confidence intervals for the model parameters. Here, $J(\widehat{\phi})$ is the total observed information matrix evaluated at $\widehat{\phi}$. Therefore, approximate $100(1-\varphi)\%$ confidence intervals for $\alpha, \beta, \lambda, a$ and b can be determined as:

$$\widehat{\alpha} \pm z_{\varphi/2}\sqrt{\widehat{J}_{\alpha\alpha}},\ \ \widehat{\beta} \pm z_{\varphi/2}\sqrt{\widehat{J}_{\beta\beta}},\ \widehat{\lambda} \pm z_{\varphi/2}\sqrt{\widehat{J}_{aa}},\ \ \widehat{a} \pm z_{\varphi/2}\sqrt{\widehat{J}_{aa}}$$

and

$$\widehat{b} \pm z_{\varphi/2}\sqrt{\widehat{J}_{bb}}$$

where $z_{\varphi/2}$ is the upper φth percentile of the standard normal distribution.

7. Application

In this section, we provide an application of the GTFr distribution to show its importance. We now provide a data analysis in order to assess the goodness-of-fit of the new model. The data set refers to brain cancer diseases in Iraq from January 1, 2009 to December 31, 2009 and it consists of 50 observations (see Al-Kanani and Jasim 2012). For this data set we shall compare the fits of the GTFr model with other competitive models namely: the Kumaraswamy transmuted Marshall–Olkin Fréchet (KTMOFr), transmuted Marshall–Olkin Fréchet (TMOFr), beta Fréchet (BFr), gamma extended Fréchet (GEFr), Marshall–Olkin Fréchet (MOFr), transmuted Fréchet (TFr) and Fréchet (Fr) distributions with corresponding densities given (for $x > 0$) by

$$\text{KTMOFr:}\, f(x) = ab\alpha\beta\theta^{\beta}x^{-\beta-1}\frac{\alpha(1+\lambda)e^{-\left(\frac{\theta}{x}\right)^{\beta}} - (\alpha\lambda+\alpha+\lambda-1)e^{-2\left(\frac{\theta}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{2a+1}}$$

$$\times\left[\alpha(1+\lambda)e^{-\left(\frac{\theta}{x}\right)^{\beta}} - (\alpha\lambda+\alpha-1)e^{-2\left(\frac{\theta}{x}\right)^{\beta}}\right]^{a-1}$$

$$\times\left\{1 - \frac{\left[\alpha(1+\lambda)e^{-\left(\frac{\theta}{x}\right)^{\beta}} - (\alpha\lambda+\alpha-1)e^{-2\left(\frac{\theta}{x}\right)^{\beta}}\right]^{a}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{2a}}\right\}^{b-1}$$

$$\text{TMOFr:}\, f(x) = \alpha\beta\theta^{\beta}x^{-\beta-1}\frac{\alpha(1+\lambda)e^{-\left(\frac{\theta}{x}\right)^{\beta}} - (\alpha\lambda+\alpha+\lambda-1)e^{-2\left(\frac{\theta}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\theta}{x}\right)^{\beta}}\right]^{2a+1}}$$

$$\text{BFr:}\, f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}\, x^{-\beta-1}\, e^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{b-1}$$

$$\text{GEFr:}\, f(x) = \frac{a\beta\alpha^{\beta}}{\Gamma(b)}\, x^{-\beta-1}\, e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{a-1}\left\{-\log\left[1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{a}\right\}^{b-1}$$

MOFr: $f(x) = \alpha\beta\theta^{\beta} x^{-\beta-1} \exp\left[-\left(\frac{\theta}{x}\right)^{\beta}\right] \left\{\alpha + (1-\alpha)\exp\left[-\left(\frac{\theta}{x}\right)^{\beta}\right]\right\}^{-2}$,

where α, β, θ, a, and b are positive parameters and $|\lambda| \le 1$.

The model selection is carried out using goodness-of-fit measures including the Akaike information criterion (AIC), consistent Akaike information criterion ($CAIC$), Hannan-Quinn information criterion ($HQIC$), Bayesian information criterion (BIC), maximized log-likelihood under the model ($-2\widehat{\ell}$), Anderson–Darling (A^*) and Cramér–Von Mises (W^*) statistics.

Table 3 lists the numerical values of goodness-of-fit statistics, whilst the MLEs and their corresponding standard errors (in parentheses) of the model parameters are given in Table 4.

The figures in Table 3 reveal that the GTFr distribution provides a superior fit to the brain cancer data than other compitive models. Figure 5 displays the fitted PDF of the GTFr distribution.

8. Conclusions

In this paper, we propose a new five-parameter distribution called the *generalized transmuted* Fréchet (GTFr) distribution, which extends the Fréchet distribution. An obvious reason for generalizing a classical distribution is the fact that the new model provides more flexibility to analyze real life data. We provide some of its mathematical and statistical properties. The GTFr density function can be expressed as a mixture of Fréchet densities. We derive explicit expressions for the ordinary and incomplete moments, generating function, residual and reversed residual life functions, order statistics and probability weighted moments. The maximum likelihood estimation of the model parameters is discussed. The proposed

Table 3. Goodness-of-fit statistics for brain cancer data.

Model	$-2\widehat{\ell}$	AIC	$CAIC$	BIC	$HQIC$	A^*	W^*
GTFr	519.669	529.669	533.309	539.229	531.033	0.050269	0.326498
KTMOFr	522.098	534.098	536.051	545.569	538.466	0.091987	0.488535
TMOFr	524.287	532.288	533.176	539.936	535.199	0.546332	3.067689
MOFr	524.609	530.609	531.131	536.346	532.794	0.160599	0.862163
GEFr	524.889	532.885	533.779	540.537	535.802	0.164100	0.870300
BFr	524.942	532.942	533.831	540.590	535.855	0.163970	0.873906
TFr	534.345	540.345	540.867	546.082	542.529	0.327316	1.811177
Fr	536.975	540.975	541.231	544.799	542.432	0.368007	2.046653

*Hold comparison between the new model and the other ones.

Table 4. MLEs and their standard errors (in parentheses) for brain cancer data.

	Estimates					
Model	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\theta}$
GTFr	20.9776977	2.5572262	−0.7140562	0.6820838	23.3572016	
KTMOFr	1.5414615	0.5775995	−0.2728668	4.1280928	34.3155264	9.7678627
TMOFr	11.2397898	2.2811016	−0.1360837			44.9213072
MOFr	13.832297	2.189104				26.221601
GEFr	43.643954	0.146632		109.608149	53.637852	
BFr	40.0087709	0.1408282		58.3679124	91.6079253	
TFr	31.6673854	1.3824328	−0.6056032			
Fr	41.151081	1.262842				

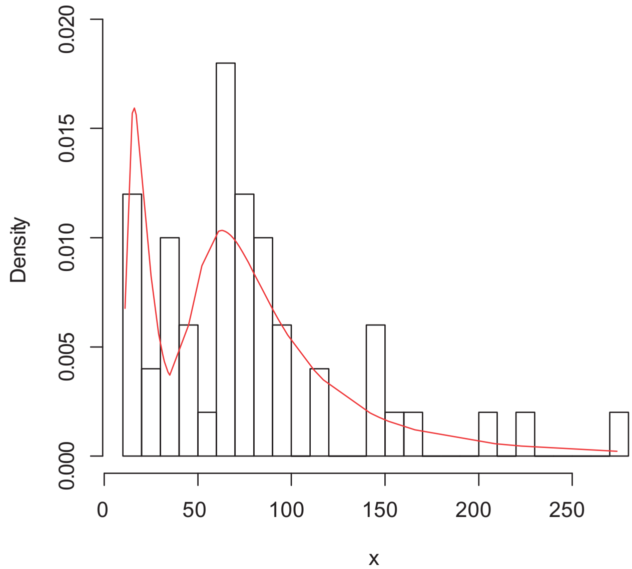

Figure 5. The estimated PDF of the GTFr model for brain cancer data.

distribution, applied to a real data set, provides better fits than some other nested and non nested models.

Acknowledgments

The authors appreciate the able assistance of a referee(s) which has greatly improved an earlier version of this paper.

References

Afify, A. Z., G. G. Hamedani, I. Ghosh, and M. E. Mead. 2015. The transmuted Marshall–Olkin Fréchet distribution: Properties and applications. *International Journal of Statistics and Probability* 4:132–48.

Afify, A. Z., H. M. Yousof, G. M. Cordeiro, Z. M. Nofal, and M. Ahmad. 2016a. The Kumaraswamy Marshall–Olkin Fréchet distribution with applications. *Journal of ISOSS* 2:41–58.

Afify, A. Z., H. M. Yousof, G. M. Cordeiro, E. M. M. Ortega, and Z. M. Nofal. 2016b. The Weibull Fréchet distribution and its applications. *Journal of Applied Statistics* 43:2608–26.

Al-Kanani, I. H., and S. A. Jasim. 2012. *Estimate survival function for brain cancer disease: On complete data by using Weibull distribution*. Saarbrücken: LAP LAMBERT Academic Publishing.

Barreto-Souza, W., G. M. Cordeiro, and A. B. Simas. 2011. Some results for beta Fréchet distribution. *Communications in Statistics—Theory and Methods* 40:798–811.

da Silva, R. V., T. A. N. de Andrade, D. B. M. Maciel, R. P. S. Campos, and G. M. Cordeiro. 2016. Anew lifetime model: The gamma extended Fréchet distribution. *Journal of Statistical Theory and Applications* 12:39–54.

Fréchet, M. 1924. Sur la loi des erreurs dIobservation. *Bulletin de la Société Mathématique de Moscou* 33:5–8.

Gupta, R. C., P. L. Gupta, and R. D. Gupta. 1998. Modeling failure time data by Lehmann alternatives. *Communictions in Statistics - Theory and Methods* 27:887–904.

Keller, A. Z., and A. R. Kamath. 1982. Reliability analysis of CNC machine tools. *Reliability Engineering* 3:449–73.

Kotz, S., and S. Nadarajah. 2000. *Extreme value distributions: Theory and applications*. London: Imperial College Press.

Krishna, E., K. K. Jose, T. Alice, and M. M. Ristić. 2013. The Marshall–Olkin Fréchet distribution. *Communications in Statistics—Theory and Methods* 42:4091–107.

Mahmoud, M. R., and R. M. Mandouh. 2013. On the transmuted Fréchet distribution. *Journal of Applied Sciences Research* 9:5553–61.

Mead, M. E., and A. R. Abd-Eltawab. 2014. A note on Kumaraswamy Fréchet distribution. *Australian Journal of Basic and Applied Sciences* 8:294–300.

Mead, M. E., A. Z. Afify, G. G. Hamedani, and I. Ghosh. 2017. The beta exponential Fréchet distribution with applications. *Austrian Journal of Statistics* 46:41–63.

Nadarajah, S., and A. K. Gupta. 2004. The beta Fréchet distribution. *Far East Journal of Theoretical Statistics* 14:15–24.

Nadarajah, S., and S. Kotz. 2003. The exponentiated exponential distribution. Retrieved from http://interstat.stat journals.net/YEAR/2003/abstracts/0312001.php.

Nofal, Z. M., A. Z. Afify, H. M. Yousof, and G. M. Cordeiro. 2017. The generalized transmuted-G family of distributions. *Communications in Statistics—Theory and Methods* 46:4119–36.

Shaw, W. T., and I. R. C. Buckley. 2007. The alchemy of probability distributions: Beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.

Treyer, V. N. 1964. Doklady Acad, Nauk, Belorus, U.S.S.R.

Yousof, H. M., A. Z. Afify, A. N. Ebraheim, G. G. Hamedani, and N. S. Butt. 2016. On six-parameter Fréchet distribution: Properties and applications. *Pakistan Journal of Statistics and Operation Research* 12:281–99.